Equivalences and Congruences on Infinite Conway Games

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The increasing use of *games* as a convenient metaphor for modeling interactions has spurred the growth of a broad variety of game definitions in Computer Science. Furthermore, in the presentation of games many related *concepts* are used, *e.g. move, position, play, turn, winning condition, payoff function, strategy, etc.* None has a unique definition. Some, but not always the same, are taken as primitive while the others are reduced to them. And many more *properties* need to be specified before the kind of game one is interested in is actually pinned down, *e.g.*: *perfect knowledge*, *zero-sum*, *chance*, *number* of players, *finiteness*, *determinacy*, *etc.* All this together with the wide gamut of games arising in real life calls for a unifying foundational approach to games. In [HL09], we started such a *programme* using very unbiased foundational tools, namely *algebras* and *coalgebras*.

We build upon Conway's notion of game. It provides an elementary but sufficiently abstract notion of game amenable to a rich algebraic-coalgebraic treatment because of the special role that *sums* of games *play* in this theory.

Conway games [Con01] are *combinatorial games*, namely no chance 2-player games, the two players being conventionally called *Left* (L) and *Right* (R). Such games have *positions*, and in any position there are rules which restrict L to move to any of certain positions, called the *Left positions*, while R may similarly move only to certain positions, called the *Right positions*. L and R move in turn, the player who plays *first* is denoted by I, while the one playing *second* is denoted by II. Notice that L or R can be either I or II and conversely. The need for this extra generality is due to the fact that in most games each player has a different set of options. Moreover, as we will see in the definition of sums of games, there may not be a strict alternation of moves in any given component.

The game is of *perfect knowledge*, *i.e.* all positions are public to both players. The game ends when one of the players has no move. In *normal* play the *other* player is the winner, while in *misère* play, the winner is the *very* player himself. The *payoff* function yields only 0 or 1. Many games played on boards are combinatorial games, *e.g. Nim*, *Domineering*, *Go*, *Chess*. Games, like Nim, where for every position both players have the same set of moves, are called *impartial*. More general games, like Domineering, Go, Chess, where L and R may have different sets of moves are called *partizan*. Many notions of games such as those which arise in Set Theory, in Automata Theory, or in Semantics of Programming Languages can be conveniently encoded in Conway's format. In [HL09], we revisit Conway's theory of terminating games and winning strategies under an *algebraic perspec-*

tive, and we introduce and study *hypergames*, *i.e.* potentially infinite games, and *non-losing* strategies, using *coalgebraic* methods. Especially in view of applications, potentially infinite, non-terminating interactions are even more important than finite ones. Traditionally, as in the automata-theoretic literature, see *e.g.* [Tho02], and denotational game semantics, [AJ94], infinite plays are taken to be winning for one of the players. Differently, we take the more natural view that all infinite plays are *draws*. Recently, this view has received attention also in the context of model checking for the μ-calculus, see *e.g.* [GLLS07].

Hypergames are defined as a final coalgebra, and operations on games can be naturally extended to hypergames, by defining them as *final morphisms*.

In the present work, we pursue further the investigation started in [HL09], by focusing on the notions of *equivalence* and *congruence*. This approach allows a unifying and perspicuous rephrasing of many results in Conways‘s theory of terminating games and winning strategies. For instance, the fact that a game has a winning strategy for the second player amounts to checking whether it is equivalent to the *empty* game. In the case of hypergames and non-losing strategies, congruences suggest the correct generalizations of the results for games. In both scenarios, focusing on equivalences allows for many intriguing results and conjectures, as shown in [HL09]. Congruences in games arise in many conceptually independent ways, and, as often happens in *semantics*, the gist of many results amounts to showing the coincidence of two congruences.

In dealing with games we can have various notions of equivalences, and hence possible congruences w.r.t. some given game constructors:

- The *final equivalence* induced by the very notion of hypergame, which abstracts superficial features of moves and hypergames.
- *Contextual equivalences* obtained by observing the *outcome* of a game, *i.e.* which player has a winning strategy, when the game is plugged in particular classes of contexts, in the style of [HL09]. This definition yields immediately a congruence, which, however, is rather difficult to establish since one is required to consider all possible contexts. Alternate definitions which use only restricted classes of basic contexts are therefore rather valuable.
- *Categorical equivalences* defined by the existence of suitable strategies, viewed as morphisms, in the style of Joyal's symmetric monoidal closed category, [Joy77]. This definition allows to establish equivalence looking only at the behaviour of a single game.
- *Order equivalences* defined through an inductively defined order relationship, in the style of Conway‘s *surreal numbers*.
- *Denotational semantical equivalences*, obtained by *interpreting* games in a subclass of canonical representatives, in the style of *Grundy numbers* for impartial games, [Gru39,Spra35]. In semantical terms, one can say that Grundy numbers provide a *fully abstract denotational semantics* to impartial games.
- *Operational semantical equivalences* obtained by defining the semantics through a *simplification*, *i.e.* re-writing process, in the style of [Con01], Theorem 69.

All the above equivalences do coincide on Conway's games, as shown in [HL09], but the situation is much more rich and intriguing in the case of hypergames.

In the present work, first we introduce and study contextual equivalences on hypergames, obtained by varying the class of contexts and the players for which we observe the existence of a winning strategy. Then we study categorical equivalences on hypergames. Since the immediate generalization of Joyal's definition to non-losing strategies does not yield a category, we introduce a somewhat weaker categorical equivalence based on the new notion of *balanced (non-losing) strategy*. Quite interestingly, this categorical equivalence captures the equivalence introduced in [BCG82], Chapter 11, on *loopy games*. We briefly discuss alternative notions of sum, all of which admit coalgebraic characterizations, and corresponding contextual equivalences.

Finally, a number of open problems and directions for future work are outlined.

References

AJ94. S. Abramsky, R. Jagadesaan. Games and Full Completeness for Multiplicative Linear logic, Journal of Symbolic Logic **59**, 1994, 543–574.

BCG82. E. Berlekamp, J. Conway, R. Guy. Winning Ways, Academic Press, 1982.

Con01. J.H. Conway. On Numbers and Games, second edition, A K Peters Ltd, 2001 (first edition by Academic Press, 1976).

GLLS07. O. Grumberg, M. Lange, M. Leucker, S. Shoham. When Not Losing Is Better than Winning: Abstraction and Refinement for the Full μ-calculus, Information and Computation **205(8)**, 2007, 1130–1148.

Gru39. P.M. Grundy. Mathematics and games, *Eureka*, **2**, 1939, 6–8.

HL09. F. Honsell, M. Lenisa. Conway Games, coalgebraically, Proc. of *CALCO'09*, Springer LNCS **5728**, 2009, 300–316, extended version available at http://sole.dimi.uniud.it/ marina.lenisa/Papers/Soft-copy-pdf/calco09extended.pdf .

Joy77. A. Joyal. Remarques sur la Theorie des Jeux a deux personnes, Gazette des sciences mathematiques du Quebec **1(4)**, 1977 (English translation by R. Houston, 2003).

Spra35. R. Sprague. Über mathematische Kampfspiele, *Tohoku Math. J.* **41**, 1935, 438–444.

Tho02. W. Thomas. Infinite games and verification, *CAV'02*, Springer LNCS **2404**, 2002, 58–64.